

APPLIED MATHEMATICS SOLVED PAPER – DECEMBER 2017

N.B:- (1) Question no. 1 is compulsory.  
(2) Attempt any 3 questions from remaining five questions.

**Q.1.(a) Evaluate  $\int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx$  [3]**

Ans : Let  $I = \int_0^{\infty} \frac{e^{-x^2}}{\sqrt{x}} dx$

Put  $x^2 = t \Rightarrow x = \sqrt{t} \Rightarrow \sqrt{x} = t^{1/4}$

Differentiate w.r.t x,

$$\therefore dx = \frac{1}{2\sqrt{t}} dt \quad \text{lim} \rightarrow [0, \infty]$$

$$\therefore I = \int_0^{\infty} \frac{e^{-t}}{1} \frac{t^{-3/4}}{2} dt$$

$$\therefore I = \frac{1}{2} \int_0^{\infty} e^{-t} \cdot t^{\frac{1}{4}-1} dt$$

But we know that,

$$\int_0^{\infty} e^{-t} \cdot t^{n-1} dt = \text{gamma}(n)$$

$$\therefore I = \frac{1}{2} \Gamma\left(\frac{1}{4}\right)$$

.....{By the definition of gamma fn}

**(b) Solve  $(D^3 + 1)^2 y = 0$  [3]**

Ans :  $(D^3 + 1)^2 y = 0$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^3 + 1)^2 = 0$$

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$$\therefore (D^3 + 1) = 0$$

Roots are :  $D = -1$  ,  $\frac{1}{2} + i\frac{\sqrt{3}}{2}$  ,  $\frac{1}{2} - i\frac{\sqrt{3}}{2}$  .. for two times

Roots of given diff. eqn are real and complex.

The general solution of given diff. eqn is given by ,

$$y_g = y_c = (c_1 + xc_2)e^{-x} + e^{\frac{x}{2}}[(c_3 + xc_4)\cos x + (c_5 + xc_6)\sin x]$$

(c) Solve the ODE  $(y + \frac{1}{3}y^3 + \frac{1}{2}x^2) dx + (x + xy^2)dy = 0$  [3]

Ans : Compare the given diff. eqn with  $Mdx + Ndy = 0$

$$\therefore M = (y + \frac{1}{3}y^3 + \frac{1}{2}x^2) \quad \therefore N = (x + xy^2)$$

$$\frac{\partial M}{\partial y} = 1 + y^2 \quad \frac{\partial N}{\partial x} = 1 + y^2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

The given differential eqn is exact.

The solution of exact differential eqn is given by,

$$\int Mdx + \int [N - \frac{\partial}{\partial y} Mdx] dy = c$$

$$\int Mdx = \int (y + \frac{1}{3}y^3 + \frac{1}{2}x^2) dx = xy + \frac{x}{3}y^3 + \frac{x^3}{6}$$

$$\frac{\partial}{\partial y} \int Mdx = x + xy^2$$

$$\int [N - \frac{\partial}{\partial y} Mdx] dy = \int [x + xy^2 - (x + xy^2)] dy = 0$$

$$\therefore xy + \frac{x}{3}y^3 + \frac{x^3}{6} = c$$

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(d) Use Taylor's series method to find a solution of  $\frac{dy}{dx} = 1 + y^2$ ,  $y(0) = 0$

At  $x=0.1$  taking  $h=0.1$  correct upto 3 decimal places. [4]

Ans :  $\frac{dy}{dx} = 1 + y^2$        $x_0 = 0, y_0 = 0, h=0.1$

$$y' = 1 + y^2 \qquad y'_0 = 1$$

$$y'' = 2yy' \qquad y''_0 = 0$$

$$y''' = 2yy'' + 2y' \cdot y' \qquad y'''_0 = 2$$

Taylor's series is given by :

$$\begin{aligned} y(0.1) &= y_0 + h \cdot y'_0 + \frac{h^2}{2!} y''_0 + \dots \\ &= 0 + 0.1(1) + \frac{0.1 \times 0.1}{2} (0) + \frac{0.1 \times 0.1 \times 0.1}{6} (2) \end{aligned}$$

$y(0.1) = 0.10033$

(e) Given  $\int_0^x \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$  using DUIS find the value of

$$\int_0^x \frac{dx}{(x^2+a^2)^2} \qquad [4]$$

Ans :  $\int_0^x \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

Differentiate w.r.t  $a$ , taking 'a' as parameter

$$\frac{d}{da} \int_0^x \frac{1}{x^2+a^2} dx = \frac{d}{da} \left[ \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \right]$$

Applying D.U.I.S rule,

D.U.I.S rule says that if function and its partial derivative is continuous then we can apply differential operator in the integral operator by converting it into partial derivative taking one parameter fro function.

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$$\int_0^x \frac{\partial}{\partial a} \frac{1}{x^2+a^2} dx = -\frac{1}{a} \tan^{-1} \frac{x}{a} \times \frac{1}{a} + \frac{-x}{a(x^2+a^2)}$$

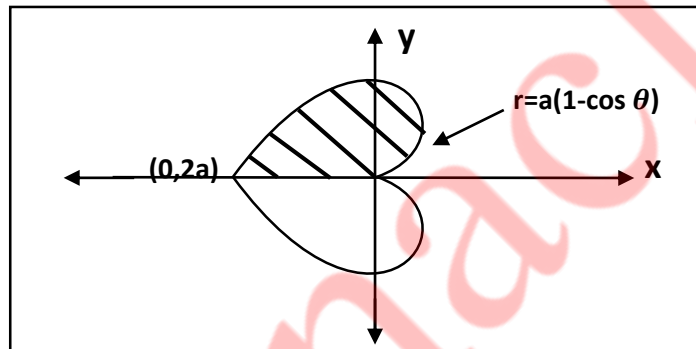
$$\int_0^x \frac{2a^2}{x^2+a^2} dx = -\frac{1}{a} \tan^{-1} \frac{x}{a} \times \frac{1}{a} + \frac{-x}{a(x^2+a^2)}$$

$$\int_0^x \frac{dx}{(x^2+a^2)^2} dx = \frac{1}{2a^3} \tan^{-1} \frac{x}{a} + \frac{x}{2a^2(x^2+a^2)}$$

(f) Find the perimeter of the curve  $r=a(1-\cos \theta)$

[4]

Ans : Curve :  $r=a(1-\cos \theta)$



Perimeter of given curve is ,

$$S = 2 \times \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

$$\frac{dr}{d\theta} = a(\sin \theta) \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = a^2 \sin^2 \theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2 [1 - 2\cos\theta + \cos^2\theta] + a^2 \sin^2\theta$$

$$\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{2}a (1 - \cos \theta)^{1/2}$$

$$= \sqrt{2}a \sqrt{2} \sin\left(\frac{\theta}{2}\right)$$

$$\therefore S = 2 \int_0^\pi \sqrt{2}a \sqrt{2} \sin\left(\frac{\theta}{2}\right) d\theta$$

$$= 4a \int_0^\pi \sin\left(\frac{\theta}{2}\right) d\theta$$

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$$= 4a \left[ -2 \cos\left(\frac{\theta}{2}\right) \right]_0^{\pi}$$

$$\therefore S = 8a$$

**Q.2.(a) Solve  $(D^3 + D^2 + D + 1)y = \sin^2 x$**

**[6]**

**Ans :**  $(D^3 + D^2 + D + 1)y = \sin^2 x$

For complementary solution ,

$$f(D) = 0$$

$$\therefore (D^3 + D^2 + D + 1) = 0$$

Roots are :  $D = -1, +i, -i$

The complementary solution of given diff eqn is ,

$$y_c = c_1 \cos x + c_2 \sin x + c_3 e^{-x}$$

For complementary solution ,

$$\begin{aligned} y_p &= \frac{1}{f(D)} X = \frac{1}{(D^3 + D^2 + D + 1)} \sin^2 x = \frac{1}{2(D^3 + D^2 + D + 1)} (1 - \cos 2x) \\ &= \frac{1}{2(D^3 + D^2 + D + 1)} e^{0x} - \frac{1}{2(D^3 + D^2 + D + 1)} \cos 2x \\ &= \frac{1}{2} - \frac{1}{2} \times \frac{1}{-D - 4 + D + 1} \cos 2x \\ &= \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{D + 1} \cos 2x \\ &= \frac{1}{2} + \frac{1}{6} \cdot \frac{1}{D + 1} \cdot \frac{D - 1}{D - 1} \cos 2x \\ &= \frac{1}{2} + \frac{1}{6} \cdot \frac{D - 1}{D^2 - 1} \cos 2x \\ &= \frac{1}{2} + \frac{1}{6} \cdot \frac{-2 \sin 2x - \cos 2x}{-4 - 1} \cos 2x \end{aligned}$$

$$y_p = \frac{1}{2} + \frac{1}{30} \cdot (2 \sin 2x + \cos 2x)$$

The general solution of given diff. eqn is given by,

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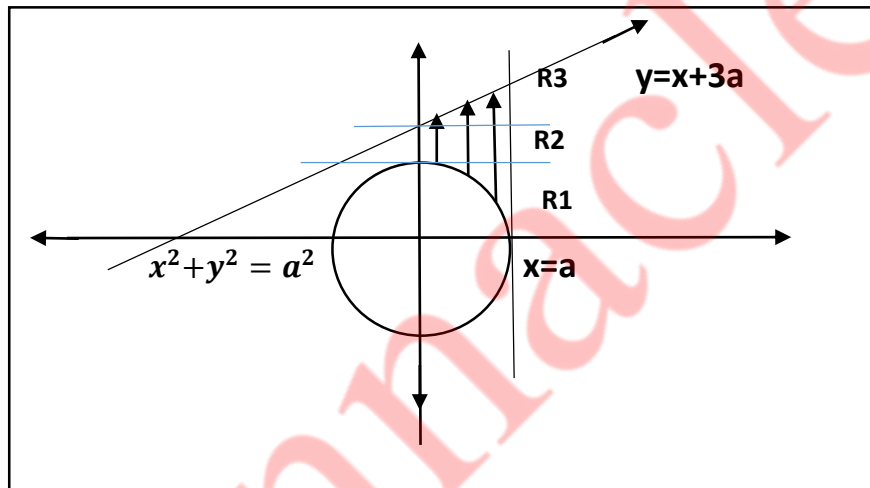
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$$y_g = y_c + y_p = c_1 \cos x + c_2 \sin x + c_3 e^{-x} + \frac{1}{2} + \frac{1}{30} \cdot (2 \sin 2x + \cos 2x)$$

(b) Change the order of integration  $\int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y) dx dy$  [6]

Ans : let  $I = \int_0^a \int_{\sqrt{a^2-x^2}}^{x+3a} f(x,y) dx dy$

Region of integration is :  $\sqrt{a^2 - x^2} \leq y \leq x + 3a$   
 $0 \leq x \leq a$



Intersection of  $x=a$  and  $y=x+3a$  is  $(a,4a)$ .

Intersection of  $x=0$  and  $y=x+3a$  is  $(0,3a)$ .

Divide the region into three parts R1,R2 and R3

$$\therefore R = R1 \cup R2 \cup R3$$

For region R1 :  $\sqrt{a^2 - y^2} \leq x \leq a$

$$0 \leq y \leq a$$

For region R2 :  $0 \leq x \leq a$

$$a \leq y \leq 3a$$

For region R3 :  $(y - 3a) \leq x \leq a$

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$$3a \leq y \leq 4a$$

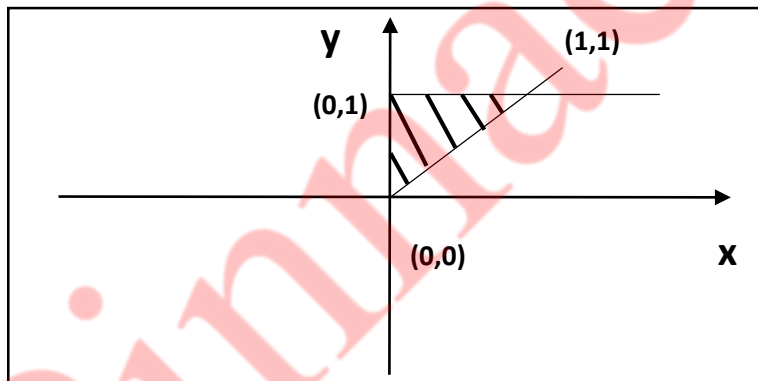
After changing the order of integration fro dydx to dxdy

$$\therefore I = \int_0^a \int_{\sqrt{a^2-y^2}}^a f(x,y) dx dy + \int_a^{3a} \int_0^a f(x,y) dx dy + \int_{3a}^{4a} \int_{(y-3a)}^{4a} f(x,y) dx dy$$

(c) Evaluate  $\int \int \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}} dx dy$ , where R is triangle whose vertices are (0,0),(1,1),(0,1). [8]

Ans : let  $I = \int \int \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}} dx dy$

Region of integration : Triangle whose vertices are (0,0),(1,1),(0,1)



The equation of lines from diagram are :  $y=1, x=y$

$$0 \leq x \leq y$$

$$0 \leq y \leq 1$$

$$\begin{aligned} \therefore I &= \int_0^1 \int_0^y \frac{2xy^5}{\sqrt{x^2y^2-y^4+1}} dx dy \\ &= \int_0^1 \int_0^y \frac{2y^5 \cdot x}{\sqrt{(1-y^4)+x^2y^2}} dx dy \end{aligned}$$

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$$\begin{aligned}
 &= \int_0^1 \int_0^y \frac{2y^5 \cdot x}{\sqrt{\frac{1-y^4}{y^2} + x^2}} \cdot \frac{1}{y} dx dy \\
 &= \int_0^1 2y^4 \left[ \sqrt{\frac{1-y^4}{y^2} + x^2} \right]_0^y dy \\
 &= \int_0^1 2y^4 \left[ \frac{1}{y} - \frac{\sqrt{1-y^4}}{y} \right] dy \\
 &= 2 \int_0^1 [y^3 - \sqrt{1-y^4} \cdot y^3] dy \\
 &= 2 \left[ \frac{y^4}{4} + \frac{1}{4} \cdot \frac{(1-y^4)^{3/2}}{3/2} \right]_0^1 \\
 &= 2 \left[ \frac{1}{4} - \frac{1}{4} \cdot \frac{2}{3} \right]
 \end{aligned}$$

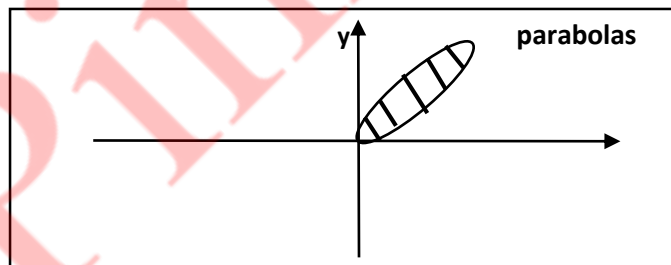
$$\therefore I = \frac{1}{6}$$

**Q.3.(a) Find the volume enclosed by the cylinder  $y^2 = x$  and  $y = x^2$**

**Cut off by the planes  $z=0, x+y+z=2$ .**

**[6]**

**Ans :** The solid is bounded by the parabolas  $y^2 = x, x^2 = y$  in the  $x y$  plane .



**In  $x$ - $y$ - $z$  plane  $x+y+z = 2$  is top base.**

**The volume between this curves is given by ,**

$$V = \iint z dx dy = \iint (2 - x - y) dx dy$$

**From the diagram we can conclude that the intersection point of both**

**Parabolas are  $(0,0), (1,1)$ .**

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$$\begin{aligned} \therefore V &= \int_0^1 \int_{x^2}^{\sqrt{x}} (2 - x - y) dx dy \\ &= \int_0^1 \left[ 2y - xy - \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx \\ &= \int_0^1 \left[ \left( 2\sqrt{x} - x\sqrt{x} - \frac{x}{2} \right) - \left( 2x^2 - x^3 - \frac{x^4}{2} \right) \right] dx \\ &= \left[ \frac{4x^{3/2}}{3} - \frac{2x^{5/2}}{5} - \frac{x^2}{4} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{10} \right]_0^1 \end{aligned}$$

$\therefore V = \frac{11}{30}$

**(b) Using Modified Eulers method ,find an approximate value of y  
At x=0.2 in two step taking h=0.1 and using three iteration**

Given that  $\frac{dy}{dx} = x + 3y$  ,  $y = 1$  when  $x = 0$ .

[6]

Ans : (i)  $\frac{dy}{dx} = x + 3y$      $x_0 = 0, y_0 = 1, h = 0.1$

$$y_1^{(0)} = y_0 + h \cdot f(x_0, y_0) = 1 + 0.1(3) = 1.3$$

$$y_1^{n+1} = y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^n)]$$

Iteration	$x_1$	$y_1^n$	$x_1 y_1^n$	$y_1^{n+1}$
0	0.1	1.3	4	1.35
1	0.1	1.35	4.15	1.3575
2	0.1	1.3575	4.1725	1.3587

$y(0.1) = 1.3587$

(ii)  $x_1 = 0.1, y_1 = 1.3587$

$$y_2^0 = 1.77631$$

$$y_2^{n+1} = y_1 + \frac{h}{2} [f(x_2, y_2) + f(x_2, y_2^n)]$$

Iteration	$x_2$	$y_2^n$	$x_2 y_2^n$	$y_2^{n+1}$
0	0.2	1.77631	5.52893	1.8439

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1	0.2	1.8439	5.7317	1.8540
2	0.2	1.8540	5.762	1.8556

$$y(0.2)=1.8556$$

(c) Solve  $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4\cos(\log(1+x))$  [8]

Ans :  $(1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4\cos(\log(1+x))$

Put  $x+1 = v \Rightarrow \frac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{dy}{dv}$$

The given eqn changes to ,

$$v^2 \frac{d^2y}{dv^2} + v \frac{dy}{dv} + y = 4\cos \log v$$

Now put  $\log v = z \therefore v=e^z$

$$[D(D-1) + D + 1]y = 4\cos z$$

$$\therefore (D^2 + 1)y = 4\cos z$$

For complementary solution ,

$$f(D) = 0$$

$$\therefore (D^2 + 1) = 0$$

Roots are : i,-i

The complementary solution of given diff. eqn is ,

$$\therefore y_c = c_1 \cos z + c_2 \sin z$$

For particular integral ,

$$y_p = \frac{1}{f(D)} X = \frac{1}{(D^2+1)} 4 \cos z = 4 \frac{z}{2} \sin z = 2 z \sin z$$

$$\therefore y_p = 2 z \sin z$$

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The general solution of given diff. eqn is given by,

$$y_g = y_c + y_p = c_1 \cos z + c_2 \sin z + 2z \sin z$$

Resubstitute  $z$  and  $v$ ,

$$y_g = c_1 \cos [\log(x + 1)] + c_2 \sin [\log(1 + x)] + 2 \log(1 + x) \sin [\log(1 + x)]$$

Q.4.(a) Show that  $\int_0^a \sqrt{\frac{x^3}{a^3-x^3}} dx = a \frac{\sqrt{\pi} \Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}$  [6]

Ans : Let  $I = \int_0^a \sqrt{\frac{x^3}{a^3-x^3}} dx$

Put  $x^3 = a^3 t \Rightarrow x = at^{\frac{1}{3}}$

Diff. w.r.t.  $x$ ,

$$dx = \frac{a}{3} t^{-2/3} dt$$

Limits becomes  $\rightarrow [0, 1]$

$$\begin{aligned} I &= \int_0^1 (t)^{3/2} \cdot (1-t)^{3/2} \cdot t^{-2/3} \frac{a}{3} dt \\ &= \frac{a}{3} \int_0^1 t^{5/6} (1-t)^{3/2} dt \\ &= \frac{a}{3} \beta\left(\frac{5}{6}, \frac{3}{2}\right) \end{aligned}$$

$$I = a \frac{\sqrt{\pi} \Gamma(\frac{5}{6})}{\Gamma(\frac{1}{3})}$$

.....{ from the definition of beta function}

(b) Solve  $(D^2 + 2)y = e^x \cos x + x^2 e^{3x}$  [6]

Ans :  $(D^2 + 2)y = e^x \cos x + x^2 e^{3x}$

For complementary solution,

$$f(D) = 0$$

$$\therefore (D^2 + 2) = 0$$

Roots are :  $D = \sqrt{2}i, -\sqrt{2}i$

Roots of given diff. eqn are complex.

The complementary solution of given diff. eqn is given by,

$$\therefore y_c = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x$$

For particular integral ,

$$\begin{aligned} y_p &= \frac{1}{f(D)} X = \frac{1}{D^2+1} e^x \cos x + \frac{1}{D^2+1} x^2 e^{3x} \\ &= e^x \frac{1}{(D+1)^2+1} \cos x + \frac{1}{D^2+1} x^2 e^{3x} \\ &= e^x \frac{1}{D^2+2D+3} \cos x + e^{3x} \frac{1}{(D+3)^2+2} x^2 \\ &= e^x \frac{1}{2} \frac{D-1}{D^2-1} \cdot \cos x + e^{3x} \frac{1}{D^2+6D+11} x^2 \\ &= e^x \frac{1}{4} (\sin x + \cos x) + \frac{e^{3x}}{11} \left[ 1 + \frac{6D+D^2}{11} \right]^{-1} x^2 \\ &= e^x \frac{1}{4} (\sin x + \cos x) + \frac{e^{3x}}{11} \left[ 1 - \frac{6D+D^2}{11} + \frac{36D^2}{121} + \dots \right] x^2 \end{aligned}$$

$$\therefore y_p = e^x \frac{1}{4} (\sin x + \cos x) + \frac{e^{3x}}{11} \left[ x^2 - \frac{12x}{11} + \frac{50}{121} \right]$$

The general solution of given diff. eqn is,

$$y_g = y_c + y_p = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x + e^x \frac{1}{4} (\sin x + \cos x) + \frac{e^{3x}}{11} \left[ x^2 - \frac{12x}{11} + \frac{50}{121} \right]$$

(c) Use polar co ordinates to evaluate  $\int \int \frac{(x^2+y^2)^2}{x^2y^2} dx dy$  over yhe area

Common to circle  $x^2+y^2 = ax$  and  $x^2 + y^2 = by, a > b > 0$ . [8]

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Ans :      Let  $I = \int \int \frac{(x^2+y^2)^2}{x^2y^2} dx dy$

Region of integration is : Area common to the circle

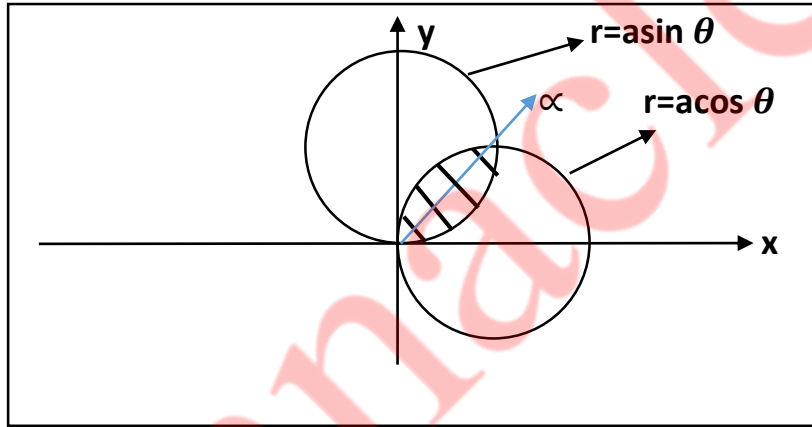
$$x^2+y^2 = ax \text{ and } x^2 + y^2 = by$$

To change the Cartesian coordinates to polar coordinates

Put  $x = r \cos \theta$     and     $y = r \sin \theta$

Circles :  $r = a \cos \theta$     and     $r = b \sin \theta$

The function becomes :  $f(x,y) = \frac{(x^2+y^2)^2}{x^2y^2} = \frac{r^4}{r^4 \sin^2 \theta \cos^2 \theta} = \frac{4}{\sin^2 2\theta} = f(r, \theta)$



Intersection of both circles is at angle  $= \tan^{-1} \frac{a}{b}$ .

Divide the region into two equal halves.

For one region ,                       $0 \leq r \leq b \sin \theta$

$$0 \leq \theta \leq \alpha$$

For another region ,                       $0 \leq r \leq a \cos \theta$

$$\alpha \leq \theta \leq \frac{\pi}{2}$$

$$\therefore I = \int_0^\alpha \int_0^{b \sin \theta} \frac{4r dr d\theta}{\sin^2 2\theta} + \int_\alpha^{\frac{\pi}{2}} \int_0^{a \cos \theta} \frac{4r dr d\theta}{\sin^2 2\theta}$$

$$\therefore I = \int_0^\alpha \frac{4}{\sin^2 2\theta} \left[ \frac{r^2}{2} \right]_0^{b \sin \theta} d\theta + \int_\alpha^{\frac{\pi}{2}} \frac{4}{\sin^2 2\theta} \left[ \frac{r^2}{2} \right]_0^{a \cos \theta} d\theta$$

$$= \frac{1}{2} b^2 \int_0^\alpha \sec^2 \theta d\theta + \frac{a^2}{2} \int_\alpha^{\frac{\pi}{2}} \operatorname{cosec}^2 \theta d\theta$$

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$$= \frac{1}{2}b^2 \tan \alpha + \frac{a^2}{2} \cot \alpha$$

$$= \frac{ab}{2} + \frac{ab}{2}$$

$$\boxed{\therefore I = ab}$$

**Q.5.(a) Solve  $ydx + x(1 - 3x^2y^2)dy = 0$**

**[6]**

**Ans :**  $ydx + x(1 - 3x^2y^2)dy = 0$  .....(1)

Compare the given eqn with  $Mdx + Ndy=0$

$$\therefore M = y$$

$$\therefore N = x(1 - 3x^2y^2)$$

$$\frac{\partial M}{\partial y} = 1$$

$$\frac{\partial N}{\partial x} = 1 - 9x^2y^2$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Hence the given diff. eqn is not exact.

But the given diff. eqn is in the form of

$$yf(xy)dx + xf(xy)dy = 0$$

$$\text{Integrating factor} = \frac{1}{Mx - Ny} = \frac{1}{xy - xy + 3x^3y^3} = \frac{1}{3x^3y^3}$$

Multiply the I.F. to eqn (1),

$$\frac{1}{3x^3y^2} dx + \left[ \frac{1}{3x^2y^3} - \frac{1}{y} \right] dy = 0$$

$$\therefore M_1 = \frac{1}{3x^3y^2} \quad N_1 = \left[ \frac{1}{3x^2y^3} - \frac{1}{y} \right]$$

Now this diff. eqn is exact.

The solution of given diff. eqn is given by,

$$\int M dx + \int \left[ N - \frac{\partial}{\partial y} M dx \right] dy = c$$

$$\int M_1 dx = \int \frac{1}{3x^3y^2} dx = \frac{-1}{6y^2x^2}$$

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$$\frac{\partial}{\partial y} \int M_1 dx = \frac{1}{3x^2y^3}$$

$$\begin{aligned} \int [N_1 - \frac{\partial}{\partial y} \int M_1 dx] dy &= \int \left[ \frac{1}{3x^2y^3} - \frac{1}{y} - \frac{1}{3x^2y^3} \right] dy \\ &= \int \frac{-1}{y} dy = -\log y \end{aligned}$$

$$\therefore \frac{-1}{6y^2x^2} - \log y = c$$

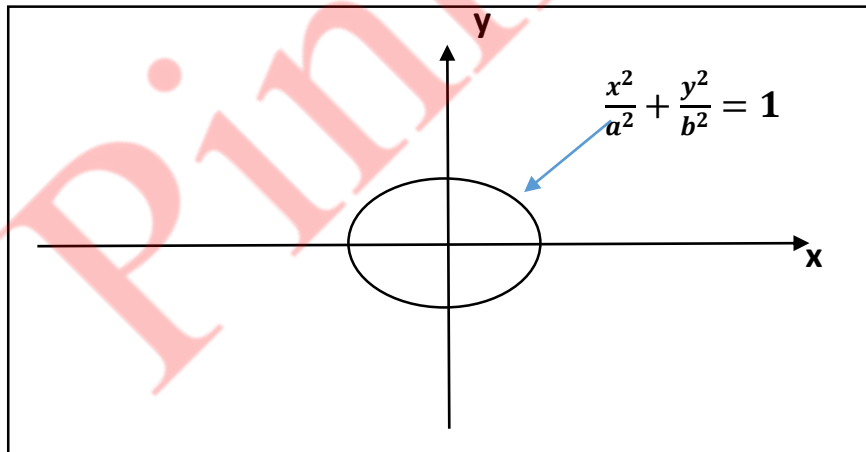
(b) Find the mass of a lamina in the form of an ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ,

If the density at any point varies as the product of the distance from the axes of the ellipse. [6]

Ans : Mass of lamina is given by ,  $M = \iint r \, dx \, dy$

r is the density function  $r = kxy$

Ellipse eqn is :  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$0 \leq y \leq b\sqrt{a^2 - x^2}/a$$

$$0 \leq x \leq a$$

$$\therefore M = 4 \int_0^a \int_0^{b\sqrt{a^2-x^2}/a} kxy \, dy \, dx$$

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$$\begin{aligned}
 &= 4k \int_0^a x \cdot \left[\frac{y^2}{2}\right] b \sqrt{a^2 - x^2} / a \, dx \\
 &= 2k \int_0^a x \cdot \frac{b^2}{a^2} (a^2 - x^2) \, dx \\
 &= \frac{2kb^2}{a^2} \int_0^a [a^2x - x^3] \, dx \\
 &= \frac{2kb^2}{a^2} \left[ \frac{a^2x^2}{2} - \frac{x^4}{4} \right]_0^a
 \end{aligned}$$

$$\therefore M = \frac{ka^2b^2}{2}$$

(c) Compute the value of  $\int_0^{\pi/2} \sqrt{\sin x + \cos x} \, dx$  using (i) Trapezoidal rule (ii) Simpson's (1/3)rd rule (iii) Simpson's (3/8)th rule by dividing into six Subintervals. [8]

Ans: Let  $I = \int_0^{\pi/2} \sqrt{\sin x + \cos x} \, dx$

Dividing limits into 6 subintervals . n=6

$a=0$  ,  $b=\frac{\pi}{2}$        $\therefore h = \frac{b-a}{n} = \frac{\pi}{12}$

$x_0 = 0$	$x_1 = \pi/12$	$x_2 = 2\pi/12$	$x_3 = 3\pi/12$	$x_4 = 4\pi/12$	$x_5 = 5\pi/12$	$x_6 = 6\pi/12$
$y_0 = 1$	$y_1 = 1.1067$	$y_2 = 1.1688$	$y_3 = 1.1892$	$y_4 = 1.1688$	$y_5 = 1.1067$	$y_6 = 1$

(i) Trapezoidal rule :  $I = \frac{h}{2} [X + 2R]$  -----(1)

$X = \text{sum of extreme ordinates} = 2$

$R = \text{sum of remaining ordinates} = 5.7402$

$I = \frac{\pi}{12 \times 2} (2 + 2(5.7402))$  .....(from 1)

$I = 1.7636$

(ii) Simpson's (1/3)<sup>rd</sup> rule :



$$I = \frac{h}{3} [ X + 2E + 4O ] \quad \text{-----(2)}$$

$$X = \text{sum of extreme ordinates} = y_0 + y_6 = 1 + 1 = 2$$

$$E = \text{sum of even base ordinates} = y_2 + y_4 = 2.3376$$

$$O = \text{sum of odd base ordinates} = y_1 + y_3 + y_5 = 3.4026$$

$$I = \frac{\pi}{3 \times 12} (2 + 2 \times 2.3376 + 4 \times 3.4026) \quad \text{.....(from 2)}$$

$I = 1.7693$

(iii) Simpson's (3/8)<sup>th</sup> rule :

$$I = \frac{3h}{8} [ X + 2T + 3R ] \quad \text{-----(3)}$$

$$X = \text{sum of extreme ordinates} = y_0 + y_6 = 0 + 0.5 = 2$$

$$T = \text{sum of multiple of three base ordinates} = y_3 = 1.1892$$

$$R = \text{sum of remaining ordinates} = y_1 + y_2 + y_4 + y_5 = 4.551$$

$$\therefore I = \frac{3 \times \pi}{8 \times 12} [ 2 + 2 \times 1.1892 + 3 \times 4.551 ]$$

$\therefore I = 1.7702$

**Q.6(a) Change the order of Integration and evaluate  $\int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4 - 4y^2}} dx dy$  [6]**

Ans : Let  $I = \int_0^2 \int_{\sqrt{2y}}^2 \frac{x^2}{\sqrt{x^4 - 4y^2}} dx dy$

Region of integration :  $\sqrt{2y} \leq x \leq 2$

$$0 \leq y \leq 2$$

Curves : (i)  $x = 2$ ,  $y = 2$ ,  $y = 0$  are lines.

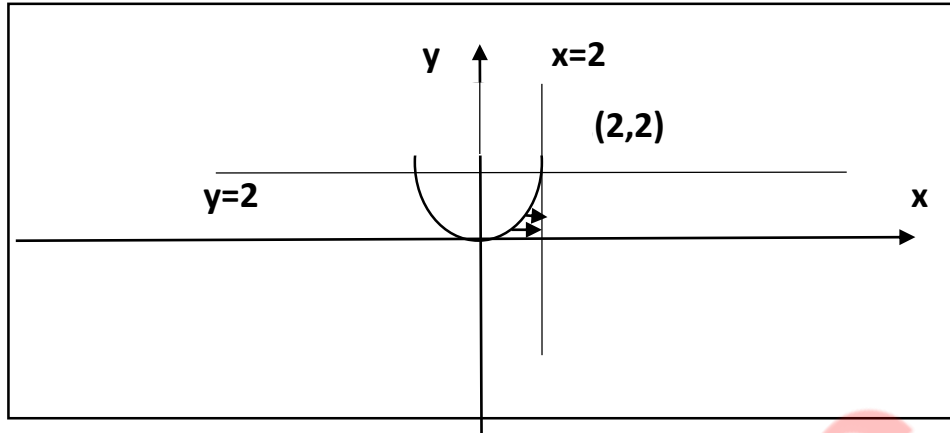
(ii)  $x = \sqrt{2y} \Rightarrow x^2 = 2y$

Parabola with vertex (0,0) opening in upward direction.

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After changing the order of integration:

$$0 \leq y \leq \frac{x^2}{2}$$

$$0 \leq x \leq 2$$

$$\therefore I = \int_0^2 \int_0^{\frac{x^2}{2}} \frac{x^2}{\sqrt{x^4 - 4y^2}} dy dx$$

$$= \frac{1}{2} \int_0^2 \int_0^{\frac{x^2}{2}} \frac{x^2}{\sqrt{\frac{x^4}{4} - y^2}} dy dx$$

$$= \frac{1}{2} \int_0^2 x^2 \left[ \sin^{-1} \left( \frac{y}{x^2/2} \right) \right]_0^{\frac{x^2}{2}} dy$$

$$\therefore I = \frac{1}{2} \int_0^2 x^2 \frac{\pi}{2} dx$$

$$= \frac{\pi}{4} \left[ \frac{x^3}{3} \right]_0^2$$

$$\therefore I = \frac{2\pi}{3}$$

(b) Evaluate  $\int \int \int x^2 dx dy dz$  over the volume bounded by planes  $x=0, y=0$

$$z=0 \text{ and } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

[8]

Ans: Let  $V = \int \int \int x^2 dx dy dz$

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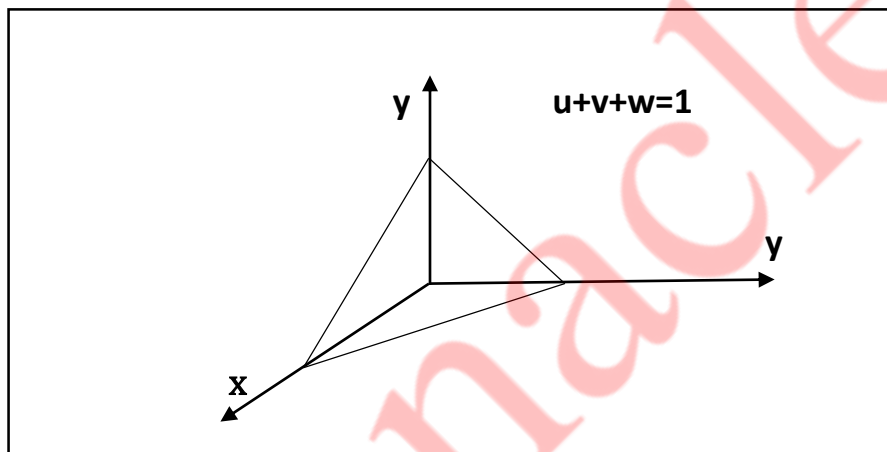
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Region of integration is volume bounded by the planes  $x=0, y=0, z=0$

And  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Put  $x = au$  ,  $y = bv$  ,  $z = cw$

$\therefore dx dy dz = abc du . dv . dw$



The intersection of tetrahedron with all axes is :  $(1,0,0), (0,1,0), (0,0,1)$ .

$$0 \leq w \leq (1 - u - v)$$

$$0 \leq v \leq (1 - u)$$

$$0 \leq u \leq 1$$

The volume required is given by ,

$$V = \int_0^1 \int_0^{1-u} \int_0^{1-u-v} abc a^2 u^2 du dv dw$$

$$= a^3 bc \int_0^1 \int_0^{1-u} (1 - u - v) u^2 dv du$$

$$= a^3 bc \int_0^1 u^2 \left[ v - uv - \frac{v^2}{2} \right]_0^{1-u} du$$

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$$= a^3 bc \int_0^1 u^2 \left[ 1 - u - u + u^2 - \frac{u^2(1-u)^2}{2} \right] du$$

$$= a^3 bc \left[ \frac{u^3}{3} - \frac{u^4}{2} + \frac{u^5}{5} - \frac{1}{2} \left( \frac{u^3}{3} - \frac{1}{2} u^4 + \frac{u^5}{5} \right) \right]_0^1$$

$$\therefore V = \frac{1}{60} (a^3 bc)$$

(c) Solve by method of variation of parameters :  $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$  [8]

Ans :  $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

For complementary solution ,

$$f(D) = 0$$

$$\therefore (D^2 - 6D + 9) = 0$$

Roots are :  $D = 3, 3$  Real roots but repeatative.

The complementary solution of given diff. eqn is ,

$$\therefore y_c = (c_1 + xc_2)e^{3x}$$

For particular solution ,

By method of variation of parameters,

$$y_p = y_1 p_1 + y_2 p_2 \quad \text{where } p_1 = \int \frac{-y_2 X}{w} dx$$

$$p_2 = \int \frac{y_1 X}{w} dx$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$w = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix} = e^{6x}$$

$$p_1 = \int \frac{-y_2 X}{w} dx = \int \frac{xe^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} dx = \int \frac{-1}{x} dx = -\log x$$

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$$p_2 = \int \frac{y_1 X}{w} dx = \int \frac{e^{3x}}{e^{6x}} \cdot \frac{e^{3x}}{x^2} dx = \int \frac{1}{x^2} dx = \frac{-1}{x}$$

The particular integral of given diff. eqn is given by,

$$\therefore y_p = -e^{3x} \log x - e^{3x} = -e^{3x}(\log x + 1)$$

The general solution of given diff. eqn is given by ,

$$y_g = y_c + y_p = (c_1 + xc_2)e^{3x} - e^{3x}(\log x + 1)$$

Pinnacle

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